Math 10A HW3 Solutions

(1)

(a) $6x - (x \cdot 2y\, dy/dx + y^2) + 6y^2 \frac{dy}{dx} = 0$

$\Rightarrow 6x - 2xy\, dy/dx - y^2 + 6y^2 \frac{dy}{dx} = 0$

$6x = 2xy\, dy/dx + y^2 - 6y^2\, dy/dx$

$-y^2 + 6x = dy/dx\,(2xy - 6y)$

$\Rightarrow dy/dx = \frac{-y^2 + 6x}{2xy - 6y}$

(b) $e^y \cos x + \sin(x) e^y \frac{dy}{dx} = 3x^2 - dy/dx$

$\Rightarrow e^y \sin(x)\, dy/dx + dy/dx = 3x^2 - e^y \cos x$

$dy/dx\,(e^y \sin(x) + 1) = 3x^2 - e^y \cos x$

$dy/dx = \frac{3x^2 - e^y \cos x}{e^y \sin(x) + 1}$

(c) $-\sin(x) = 2y\, dy/dx + 3x^2$

$\Rightarrow 2y\, dy/dx = -\sin(x) - 3x^2$

$dy/dx = \frac{-\sin(x) - 3x^2}{2y}$

(2)

(a) $xy = 25$; $(5,5)$

$\Rightarrow x \cdot \frac{dy}{dx} + y \cdot 1 = 0$

$x \cdot \frac{dy}{dx} = -y \Rightarrow \frac{dy}{dx} = \frac{-y}{x}\Big|_{(5,5)} = -1$

So we get $y - 5 = -1(x - 5)$

(b) $1/x = 1/y = 1/2$; $(6,3)$

$\Rightarrow \frac{1}{x} - \frac{1}{y} = \frac{1}{2}$ or $x^{-1} - y^{-1} = \frac{1}{2}$

$\Rightarrow -x^{-2} + y^{-2}\frac{dy}{dx} = 0$

$\Rightarrow \frac{-1}{x^2} + \frac{dy/dx}{y^2} = 0$

$\Rightarrow \frac{dy}{dx}\left(\frac{1}{y^2}\right) = \frac{1}{x^2}$

$\Rightarrow \frac{dy}{dx} = \frac{1/x^2}{1/y^2} = \frac{y^2}{x^2}\Big|_{(6,3)} = \frac{9}{36} = \frac{1}{4}$

So $y - 3 = \frac{1}{4}(x-6)$.

(c) $y^2(6-x) = x^3$; $(3,3)$

$\Rightarrow 6y^2 - xy^2 = x^3$

$\Rightarrow 12y\frac{dy}{dx} - \left(x \cdot 2y\frac{dy}{dx} + y^2 \cdot 1\right) = 3x^2$

$\Rightarrow 12y\frac{dy}{dx} - 2xy\frac{dy}{dx} - y^2 = 3x^2$

$\frac{dy}{dx}(12y - 2xy) = 3x^2 + y^2$

$\frac{dy}{dx} = \frac{3x^2+y^2}{12y-2xy}\Big|_{(3,3)} = \frac{3\cdot 9 + 9}{12\cdot 3 - 2(3\cdot 3)} = \frac{36}{18} = 2$

So we get $y-3=2(x-3)$.

(3)

(a) $f(x)=e^x e^2$

$f'(x)=e^2e^x$

$f''(x)=e^2e^x$

(b) $f(x)=\cos(3x-1)$

$f'(x)=-\sin(3x-1)\cdot 3=-3\sin(3x-1)$

$f''(x)=-3\cos(3x-1)\cdot 3=-9\cos(3x-1)$

(c) $f(x)=\sin(x)-\ln(x^2)$

$f'(x)=\cos x-\left(\frac{1}{x^2}\cdot 2x\right)=\cos x-2x^{-1}$

$f''(x)=-\sin x+2x^{-2}$

(4) $f(t)=\sin(\pi t)$ gives position @ t

$f'(t)=\pi\cos(\pi t)$ gives velocity @ t

$f''(t)=-\pi^2\sin(\pi t)$ gives acceleration @

$\Rightarrow f''(3)=0$

(5) f increasing $(-2,-1)$, $(1,+\infty)$
f decreasing $(-\infty,-2)$, $(-1,+1)$
critical points at $x = -2, -1, 1$

local min local max local min

(6) I, E (max @ $x=0$, look for derivative graph with zero @ $x=0$)

II, F (look for derivative graph with 2 zeros)

III, B (look for derivative graph with 3 zeros)

IV, A (look for derivative graph with 4 zeros)

V, D (look for derivative graph with no zeros)

VI, C (look for derivative graph with zero @ $x=-1.5$)

(7)

(a) $\sin(x)$; $(-\pi, \pi)$

$f'(x) = \cos x = 0$

critical points $x = -\pi/2, \pi/2$

$-$ $+$ $-$

$-\pi/2$ $\pi/2$

local max @ $x = \pi/2$

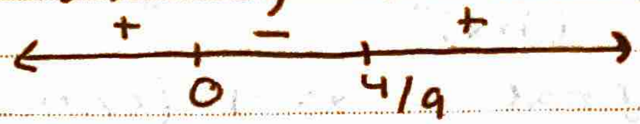
local min @ $x = -\pi/2$

(b) $f(x) = 3x^3 - 2x^2$; $(-2, 2)$

$f'(x) = 9x^2 - 4x = 0$

$x(9x - 4) = 0$

$\Rightarrow x = 0, 4/9$ critical points



local max @ $x = 0$

local min @ $x = 4/9$

(c) $f(x) = e^x$; $(0, \infty)$

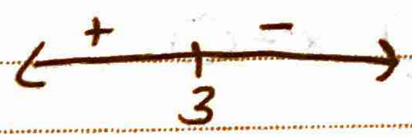
$f'(x) = e^x = 0 \rightsquigarrow$ no x satisfies eqt'n

no critical points

(d) $f(x) = 10 + 6x - x^2$; $(-\infty, +\infty)$

$f'(x) = 6 - 2x = 0$

$x = 3$ critical point

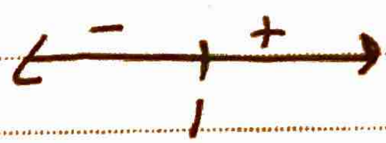


local max @ $x = 3$

(e) $f(t) = e^{t^2 - 2t + 1}$; $(-\infty, +\infty)$

$f'(t) = e^{t^2 - 2t + 1} \cdot (2t - 2) = 0$

$t = 1$ critical point



local min @ $t = 1$

(10) False, consider $f(x)=x^2$ on $(-10,0)$ where $f'(x)=2x$ and $f''(x)=2$.

(11) True!

(12)

(a) $f(x) = x/1+x^2$

$\Rightarrow f'(x) = \dfrac{(1+x^2)-x(2x)}{(1+x^2)^2}$

$f'(x) = \dfrac{1+x^2-2x^2}{(1+x^2)^2} = \dfrac{1-x^2}{(1+x^2)^2} = 0$

$\Leftrightarrow x = \pm 1$

$-\quad +\quad -$

$-1 \qquad 1$

local min $\qquad$ local max

$f''(x) = \dfrac{(1+x^2)^2(-2x)-(1-x^2)2(1+x^2)(2x)}{(1+x^2)^4}$

$= \dfrac{-2x(1+x^2)^2-4x(1+x^2)(1-x^2)}{(1+x^2)^4}$

$= \dfrac{-2x(1+x^2)-4x(1-x^2)}{(1+x^2)^3} = \dfrac{-2x-2x^3-4x+4x^3}{(1+x^2)^3}$

$f''(x) = \dfrac{2x^3-6x}{(1+x^2)^3} = 0$

$\Leftrightarrow 2x^3-6x=0$

$\Leftrightarrow 2x(x^2-3)=0$

$\Leftrightarrow x = 0, \pm\sqrt{3}$

$-\quad +\quad -\quad +$

$-\sqrt{3} \qquad 0 \qquad \sqrt{3}$

(12) (a) continued

$f(0) = 0$

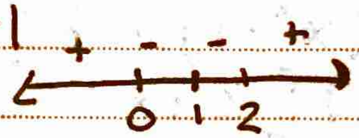
(b) $f(x) = (x-1)^{-1} + x$

$f'(x) = -(x-1)^{-2} + 1 = {}^{-1}/(x-1)^2 + 1$

$f'(x) = {}^{-1}/(x-1)^2 + 1 = 0$

$\Leftrightarrow {}^{-1}/(x-1)^2 = -1$

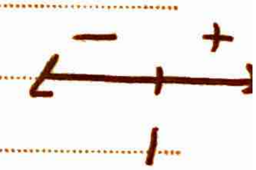
$\Leftrightarrow x = 0, 2$



also undefined if $x=1$

$f''(x) = 2(x-1)^{-3} = {}^{2}/(x-1)^3 \neq 0$

undefined if $(x-1)^3 = 0 \Leftrightarrow x=1$



vertical asymptote at $x=1$, oblique asymptot

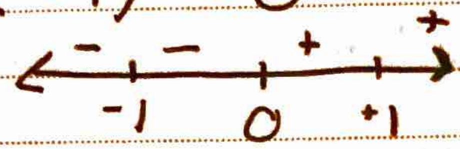
Also see that $f(0) = -1$

(c) $f(x) = (x^2-1)^3$

$f'(x) = 3(x^2-1)^2 \cdot (2x) = 6x(x^2-1)^2$

$f'(x) = 6x(x^2-1)^2 = 0$

$\Leftrightarrow \quad x = 0, \pm 1$



$f''(x) = 6x \cdot (2(x^2-1)(2x)) + (x^2-1)^2 \cdot 6$

$f''(x) = 24x^2(x^2-1) + 6(x^2-1)^2$

$f''(x) = (x^2-1)[24x^2 + 6(x^2-1)]$

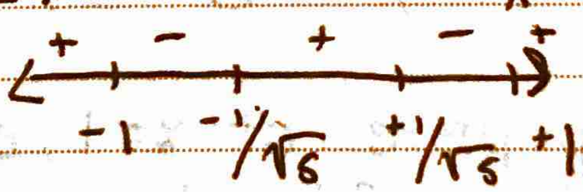
$= (x^2-1)[24x^2 + 6x^2 - 6]$

$f''(x) = (x^2-1)[30x^2 - 6] = 0$

$\Leftrightarrow \quad x^2 - 1 = 0 \quad$ or $\quad 30x^2 - 6 = 0$

$x^2 = 1 \qquad x^2 = 6/30 = 1/5$

$x = \pm 1 \qquad x = \pm\sqrt{1/5} = \pm 1/\sqrt{5}$



We also see that:

$f(0) = (-1)^3 = -1$

$f(x) = (x^2-1)^3 = 0 \quad \Leftrightarrow \quad x = \pm 1$

(d) $f(x) = x^{1/5}(4-x) = 4x^{1/5} - x^{6/5}$

$f'(x) = \frac{4}{5}x^{-4/5} - \frac{6}{5}x^{1/5} = 0$

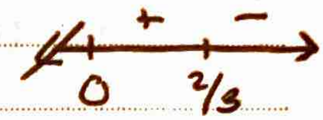
$\Leftrightarrow f'(x) = 4x^{-4/5} - 6x^{1/5} = 0$

$\Leftrightarrow \dfrac{4-6x}{x^{4/5}} = 0$

$\Leftrightarrow 4 - 6x = 0$ or $x = 2/3$

undefined if $x = 0$

(recall $x \geq 0$ for f to be defined)



$f''(x) = -\frac{16}{25}x^{-9/5} - \frac{6}{25}x^{-4/5}$ concave down

Notice also that :

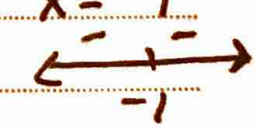
$f(0) = 0$

$f(x) = x^{1/5}(4-x) = 0 \Leftrightarrow x = 0, 4$

(e) $f(x) = \frac{2+x}{1+x}$ (undefined if $x = -1$)

$f'(x) = \dfrac{(1+x)(1) - (2+x)(1)}{(1+x)^2} = \dfrac{1+\cancel{x}-2-\cancel{x}}{(1+x)^2} = \dfrac{-1}{(1+x)^2}$

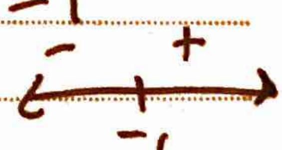
$f'(x)$ undefined if $(1+x)^2 = 0 \Leftrightarrow x = -1$



$f''(x) = \frac{d}{dx}\left(-(1+x)^{-2}\right)$

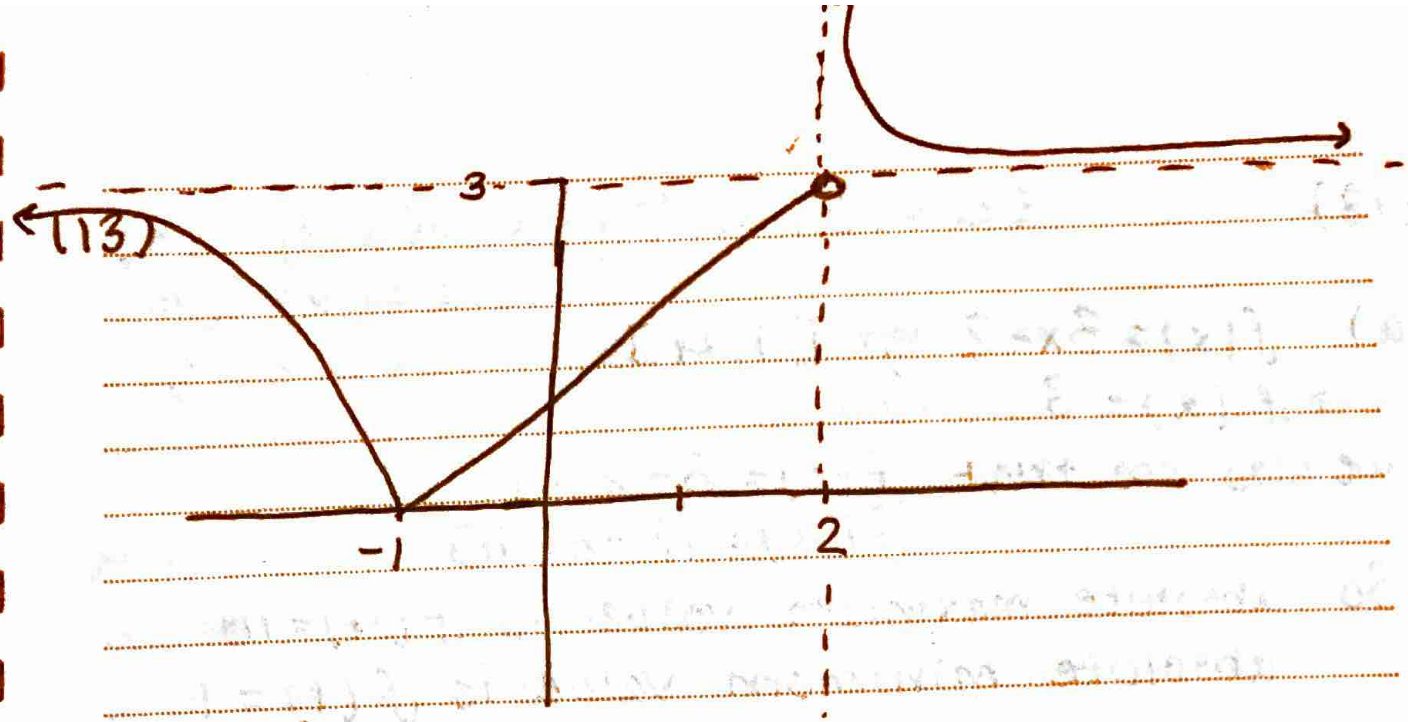
$f''(x) = 2(1+x)^{-3}$

undefined if $(1+x)^3 = 0 \Leftrightarrow x = -1$

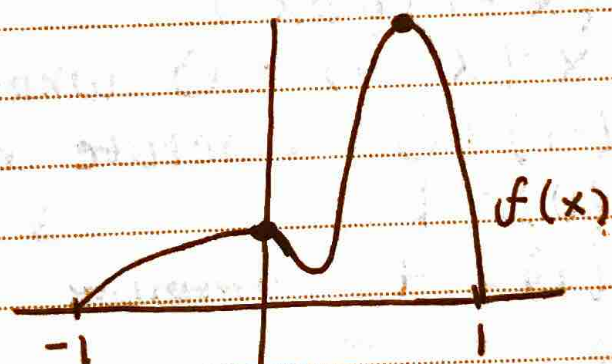


Notice also that:

- vertical asymptote at $x=-1$
- horizontal asymptote at $y=1$
- $f(0)=2$
- $f(x)=\dfrac{2+x}{1+x}=0 \Leftrightarrow x=-2$



(14) False, consider :



(15) False, let $f(x)=x^2$ on $[1, 10]$

(16) True using #11

(17) True !

(18)

(a) $f(x) = 3x - 2$; $[1, 4]$

$\Rightarrow f'(x) = 3$

we also see that $f(1) = 3 - 2 = 1$

$f(4) = 12 - 2 = 10$

So absolute maximum value is $f(4) = 10$

absolute minimum value is $f(1) = 1$

(b) $f(x) = (x^3 + 1)^2$; $[-1, 1]$

$\Rightarrow f'(x) = 2(x^3 + 1)(3x^2)$

$f'(x) = 6x^2(x^3 + 1) = 0$ when $x = 0, -1$

We see that $f(-1) = 0$ absolute minimum valu

$f(0) = 1$

$f(1) = 4$ absolute max value

(c) $f(x) = \cos(\pi x)$; $[-1/2, +1/2]$

$\Rightarrow f'(x) = -\pi \sin(\pi x) = 0$ when $x = 0$

We see that $f(-1/2) = 0$ absolute min value

$f(0) = 1$ absolute max value

$f(1/2) = 0$ absolute min value

(d) $f(x) = |x|$; $[-5, 5]$

* f not differentiable at $x = 0$!

For $-5 \le x < 0$, $f(x) = -x \Rightarrow f'(x) = -1$

For $0 < x \le 5$, $f(x) = x \Rightarrow f'(x) = 1$

$f(0) = 0$ absolute min value

$f(-5) = f(5) = 5$ absolute max value

(e) $f(x) = x + x^{-1}$; $(0, +\infty)$

$\Rightarrow f'(x) = 1 - x^{-2}$

$f'(x) = 1 - 1/x^2 = 0 \iff 1 = 1/x^2$

$\iff x = \cancel{\pm}1$

Notice there is vertical asymptote at $x = 0$.
We have global minimum at $x = 1$ with absolute min. value $f(1) = 2$.

(f) $f(x) = xe^{-x}$; $(-\infty, +\infty)$

$\Rightarrow f'(x) = x \cdot (-e^{-x}) + (e^{-x}) \cdot 1$

$f'(x) = -xe^{-x} + e^{-x} = 0$

$\iff e^{-x}(1 - x) = 0$

$\iff x = 1$

We have an absolute max at $x = 1$ with abs. max value $f(1) = 1e^{-1} = 1/e$

(19) $x + y = 75 \longrightarrow y = 75 - x$

Want to maximize $G(x) = x(75 - x)^2$.
Can rewrite $G(x) = x^3 - 150x^2 + 5625x$
so $G'(x) = 3x^2 - 300x + 5625 = 0$

$\iff x = 25, \cancel{75}$ so $y = 50$

(20) $2w + 2\ell = 100$

$\Rightarrow 2w = 100 - 2\ell$

$w = 50 - \ell$

Want to maximize $A(\ell) = \ell(50 - \ell) = 50\ell - \ell^2$

$A'(\ell) = 50 - 2\ell = 0 \Leftrightarrow 50 = 2\ell$

$\Leftrightarrow \ell = 25$ so $w = 25$

(21) $P(t) = {6000t}/{60 + t^2}$

$P'(t) = \dfrac{(60 + t^2)(6000) - 6000t(2t)}{(60 + t^2)^2}$

$P'(t) = \dfrac{6000(60 + t^2 - 2t^2)}{(60 + t^2)^2} = \dfrac{-6000(t^2 - 60)}{(t^2 + 60)^2}$

$P'(t) = 0$ when $t = \sqrt{60}$ days

(22) $x + r = 100 \rightarrow x = 100 - r$

want to maximize $G(r) = (100 - r)^2 + \pi r^2$

$\rightarrow G'(r) = -2(100 - r) + 2\pi r = 0$

$\Leftrightarrow r = {100}/{\pi + 1}$

So the pieces should be $\frac{100}{\pi + 1}$ cm long &

$100 - \left(\frac{100}{\pi + 1}\right)$ cm long.

(23) The distance between point $(0,0)$ and (x,y) is given by:

$$d = \sqrt{x^2 + y^2}$$

but $y = 1-x^2$ so we have

$$d = \sqrt{x^2 + (1-x^2)^2} \quad \text{or} \quad d^2 = x^2 + (1-x^2)^2.$$
$$= f(x)$$

Notice $f'(x) = 2x + 2(1-x^2)(-2x)$

$$f'(x) = 2x - 4x(1-x^2)$$
$$= 2x - 4x + 4x^2$$
$$= -2x + 4x^2$$
$$= 2x(2x-1) = 0$$

$$\Longleftrightarrow x = \cancel{0}, \tfrac{1}{2}$$

$+ \quad - \quad +$

$0 \quad \tfrac{1}{2}$

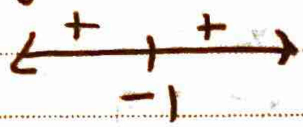
So the point on $y = 1-x^2$ closest to origin is $(1/2, 3/4)$.

(f) $f(x) = x/1+x$; $(-\infty, +\infty)$

$f'(x) = \frac{(1+x)\cdot(1) - (x)(1)}{(1+x)^2} = \frac{1+x-x}{(1+x)^2}$

$f'(x) = \frac{1}{(1+x)^2}$

undefined if $x=-1$ (critical point)



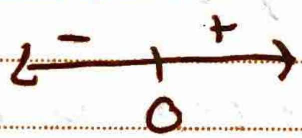
no local max/min

(g) $f(x) = 1 - e^{-x^2}$; $(-\infty, +\infty)$

$f'(x) = -e^{-x^2} \cdot (-2x)$

$f'(x) = 2xe^{-x^2} = 0$

$x=0$ critical point



local min @ $x=0$

(8) $f''(0)=0$

2nd derivative test inconclusive

~~no sign change~~

$x=0$ is neither a local max/min since no sign change